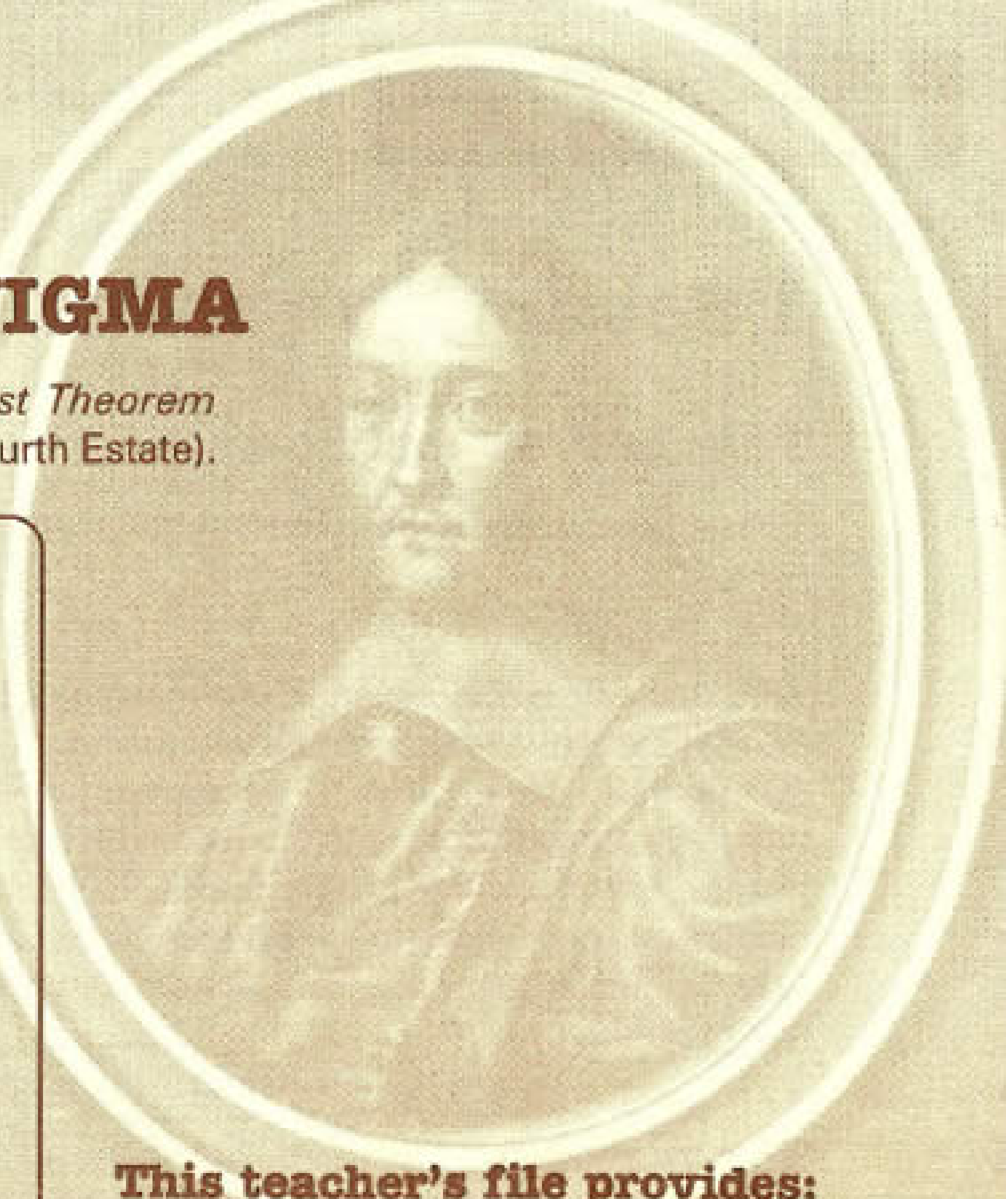
FERMAT'S ENIGMA

Based on the book *Fermat's Last Theorem* by Simon Singh (published by Fourth Estate).



Pure mathematicians just love a challenge. They love unsolved problems. When doing maths there is this great feeling. You start with a problem that mystifies you. You can't understand it, it's so complicated, you just can't make head or tail of it. But then when you finally resolve it, you have an incredible feeling of how beautiful it is, how it all fits together so elegantly. Most deceptive are the problems that look easy and yet turn out to be extremely intricate. Fermat's last theorem is the most beautiful example of this. It just looked as though it had to have a solution, and of course it is very special because Fermat said that he had a solution.

—Andrew Wiles

This teacher's file provides:

- A series of lessons—for 13 to 15 year olds—aimed at helping students see mathematics as a living subject full of beauty, order and power.
- An exposure to mathematicians who enjoyed discovering patterns and had a passion for finding truth.
- An opportunity to look at some topics in number theory (prime numbers, Pythagorean triples) and at the idea of a mathematical proof at some depth.

Students should have studied the Pythagoras theorem and be familiar with rational and irrational numbers.

The lessons and activities outlined may be introduced to a mixed-ability class. Students should be encouraged to memorise the first thirty square numbers.

With encouragement, all students should be able to handle the activities given and a few may be challenged to take up the more advanced extensions suggested.

Further readings from the book, which would enrich the content of the lessons, are also suggested.

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1 Setting the scene

Teacher exposition

'I have discovered a truly marvelous proof which this margin is too narrow to contain'

—Fermat

With this enigmatic jotting in the margin of Diophantus's *Arithmetica*, Pierre de Fermat set a challenge that would torment the finest mathematicians for almost 350 years. It soon came to be known as Fermat's last theorem. For centuries this is the only theorem of Fermat which persistently failed to yield to a proof. Yet the beauty of the problem was its downright simplicity—it was a problem that could be understood by any high school student. The suggestion that Fermat had actually found a solution increased manifold the interest level in the problem.

The theorem

$X^n + Y^n = Z^n$ has no whole number solution for $n>3$, and where X,Y and Z are whole numbers.

Discuss

How many solutions are there for $n = 1$, $n = 2$?

Students will notice that the first case is a trivial one and there are an infinite number of solutions.

Most of them should recognize the second case as the Pythagoras theorem, found in all school textbooks at class 7 or 8 level.

Individual activity

How many solutions can you find for $X^2 + Y^2 = Z^2$ where X, Y, Z have no common factors?

List out as many as you can.

These are known as the primitive Pythagorean triples.

Investigation

Pythagorean triples

Look at the triangles in the box carefully. They all belong to a particular family of primitive Pythagorean triples .

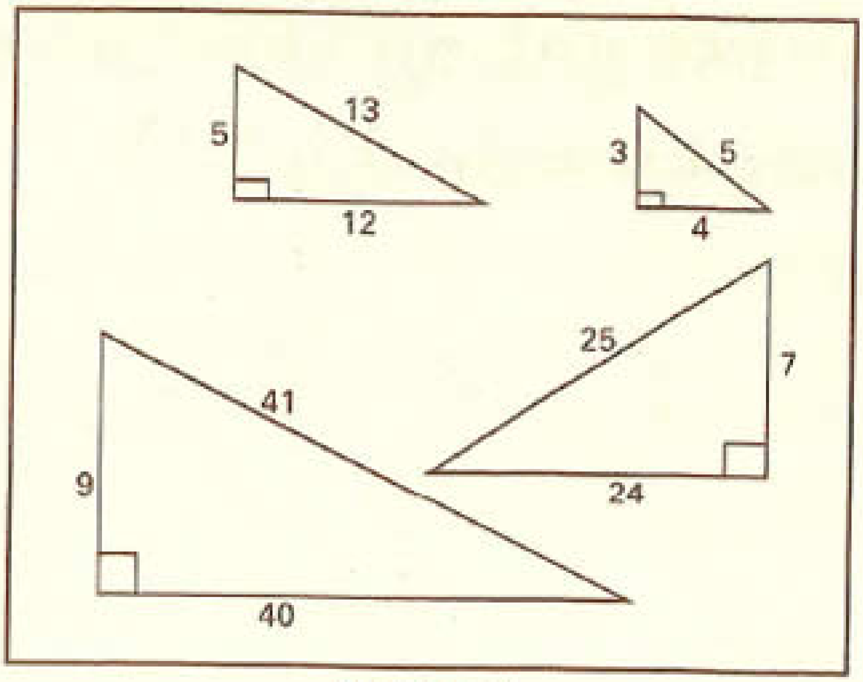


Diagram 1

- Investigate the relationships between the lengths of the sides of the triangles that belong to this set.
- Use this relationship to generate a few more Pythagorean triples.
- Can you show that there are an infinite number of such triples?

  (Hint: Look at the difference between consecutive square numbers. How many odd numbers are there which are also square numbers?).
- Investigate rules for finding the perimeter and area of triangles that belong to this set, when you know the length of the shortest side.

Note

One of the interesting aspects of the last theorem is that when one looks at the next degree up ($n = 3$) one discovers that there are no solutions at all—one goes from an infinite number of solutions (for $n = 2$) to zero solutions by going a single degree up.

Possible extensions

- Research Greek mathematicians.
- Pythagoras and the irrational numbers.
- Proof of the method for generating Pythagorean triples.

2 Investigating the nature of a mathematical truth

Discussion

How can we be sure that the Pythagorean identity, $X^2 + Y^2 = Z^2$, is true for all right-angled triangles?

Individual Activity

Prove the Pythagoras theorem using the diagram below.

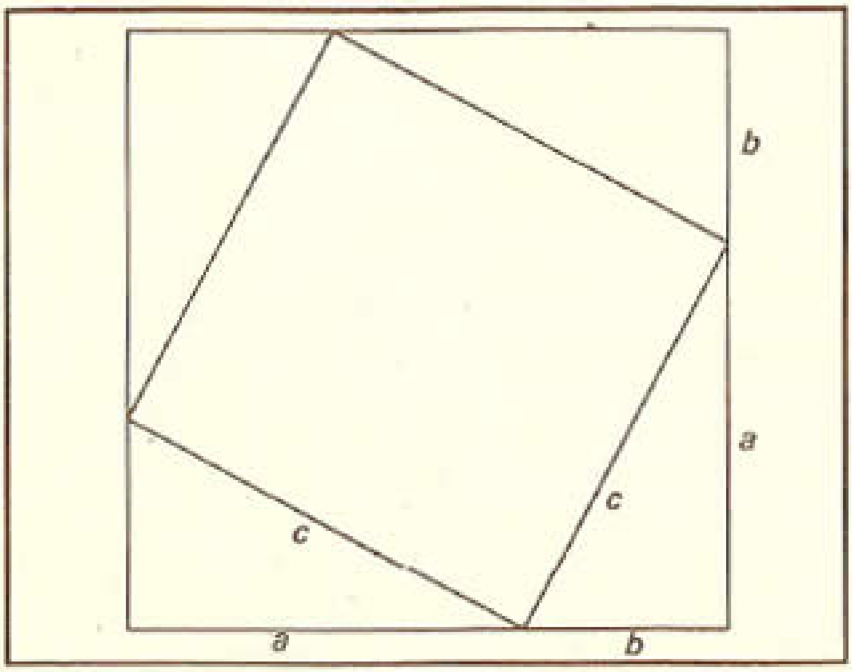


Diagram 2

Teacher exposition

The concept of absolute proof, which leaves no room for ambiguity lies at the heart of mathematics and distinguishes it from the other sciences. The Pythagoras theorem, proved 2,500 years ago, is as valid today as it was then, whereas most of our theories in science have undergone a change over time. In science a theory or hypothesis is accepted as being valid as long as the empirical data fits *favourably* with its predictions. The requirement for a mathematical proof is a lot more rigorous—the conjecture has to be true for all cases—and it should be demonstrated using logic. Even after the advent of the computer with its huge number crunching capacity, though Fermat's theorem could be verified to be true for values of n upto four million, mathematicians would still not claim that the theorem had been proved, for there remained a possibility of coming upon a counter example to disprove the theorem.

Euclid was a Greek who lived around 300 BC. He was the first mathematician to systematically use the notion of proof to arrive at a theorem. He stated:

- Begin with a series of axioms, statements that can be assumed to be true or are self-evidently true.
- Make a reasonable conjecture about some mathematical relationship that appears to be true (a hunch backed up by some evidence).
- Proceed by a step-by-step argument and arrive at a conclusion which is then called a theorem.

Euclid's seminal work, *Elements,* is based on the above notion of proof.

An example of Euclid's approach is the theory of infinity of primes. Mathematicians before Euclid had suspected that there was an infinite number of prime numbers. Euclid actually proved that this is so. What had been a guess was established as a fact, beyond any doubt.

Euclid proved that there are an infinite number of primes in the following way:

- Assume that there are a finite number of primes $p_1, p_2, p_3 \ldots \ldots p_k$ where p_k is the largest prime.
- Multiply all the k prime numbers and then add 1. Call the answer P; $P = (p_1 \times p_2 \times p_3 \ldots \ldots p_k) + 1$.
- This number P is not divisible by any of the primes $p_1, p_2, p_3 \ldots \ldots p_k$ which implies that P is a prime number.
- The largest prime is p_k but P is larger than p_k which is absurd, and therefore our original statement must be false.
- Therefore there must be an infinite number of primes.

This method of proof is called *Reductio ad absurdum* : assuming the opposite of what you want to prove and then showing that this leads to something absurd—a contradiction. It is also known as proof by contradiction.

Another powerful form of proof is the method of proof by induction. It is essentially a two-step process:

- Prove that the statement is true for the first case.
- Prove that if the statement is true for any one case then it is also true for the next case.

It is like trying to topple an infinite number of carefully arranged dominoes by toppling the first domino.

Extension activity

- Prove Ö 2 is irrational using the method of proof by contradiction
- Prove $p^2 - 1$ is divisible by 24, where p is a prime greater than 3.

(Box 2 would be useful for conducting the second activity as a classroom exercise)

Working at a proof

Divide the class into groups of four. There are four hints cards which should be given roughly at 10 minute intervals. This activity should be carried out with minimal teacher intervention.

Prove that $p^2 - 1$ is always a multiple of 24 where p is a prime number greater than 3.

Hint Card 1

- Factorize $p^2 - 1$
- 1,2,3,4 ....(p-1), p, (p + 1)........
- What factors do (p-1) and (p + 1) have? Remember p is prime and so 2 and 3 cannot be factors of p

Hint Card 2

- P must be an odd number, so (p-1) and (p + 1) must both be ...
- Every other even number is a multiple of....
- So either (p-1) or (p+1) must be....

Hint Card 3

- (p-1), p, (p+1) are consecutive numbers, so one of them must be a multiple of....
- p is not a multiple of 3, so......

Hint Card 4

- Either (p-1) is a multiple of 3 or (p+1) is a multiple of 3
- Either (p-1) is a multiple of 4 or (p+1) is a multiple of 4
- So one of them is a multiple of 3 and the other a multiple of 4 and both of them are multiples of 2

Therefore (p-1)(p+1) must be......

3 Fermat

Teacher exposition

A brief history of Fermat's life.

What kind of a person was Fermat? Where did he get his inspiration to pose such a problem?

He was born at the turn of the 17th century (20 August, 1601). There is no record of the young Fermat excelling in mathematics. His family steered him towards a career in the civil service. But he devoted all his spare time to mathematics. Together with Pascal, he developed probability theory. He is also considered to be the pioneer of Calculus, which Newton later developed into a full-fledged discipline. However his greatest passion was for Number theory where he created numerous theorems, one of which was to go down in history as the most famous theorem.

It seems that the *Arithmetica* by Diophantus—who lived around 250 BC in Alexandria—was a great inspiration for Fermat. Diophantus collected well-understood problems and created new ones and compiled them all in a major treatise entitled *Arithmetica.* His speciality was in creating problems that required inte solutions: today these are referred to Diophontine problems. During the turmoil the Dark Ages only six of his thirteen boo survived. The theory of numbers, which h stood still since the barbaric burning Alexandria, was now given a new impetus Fermat.

Riddle on Diophantus' tomb

God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, he clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage he granted him a son. Alas! Late-born wretched child after attaining the measure of half his father's full life, chill fate took him. After consoling his grief by this science of numbers for four years he ended his life.

How long did he live?

Answer: 84 years

4 Euler (B. 1707)

Teacher exposition

Leonard Euler, regarded as the best mathematician of the 18th century, was the first person to make progress on proving Fermat's last theorem after its publication in 1670. Fermat had cryptically described a proof for the specific case of n = 4 using a form of proof by contradiction known as the method of infinite descent. In 1753, after almost 80 years, Euler announced his proof for the case n = 3. In this he had to incorporate the use of *imaginary numbers*. Unfortunately his efforts to find a general proof all ended in failure—the man who created more mathematics than anybody else was humbled by Fermat's challenge. Nevertheless he made an immense contribution to the development of mathematics in the long established branches of number theory, analysis, algebra and geometry. He also ventured into the largely unexplored territory of graph theory and differential geometry. He created a new branch of mathematics known as network theory.

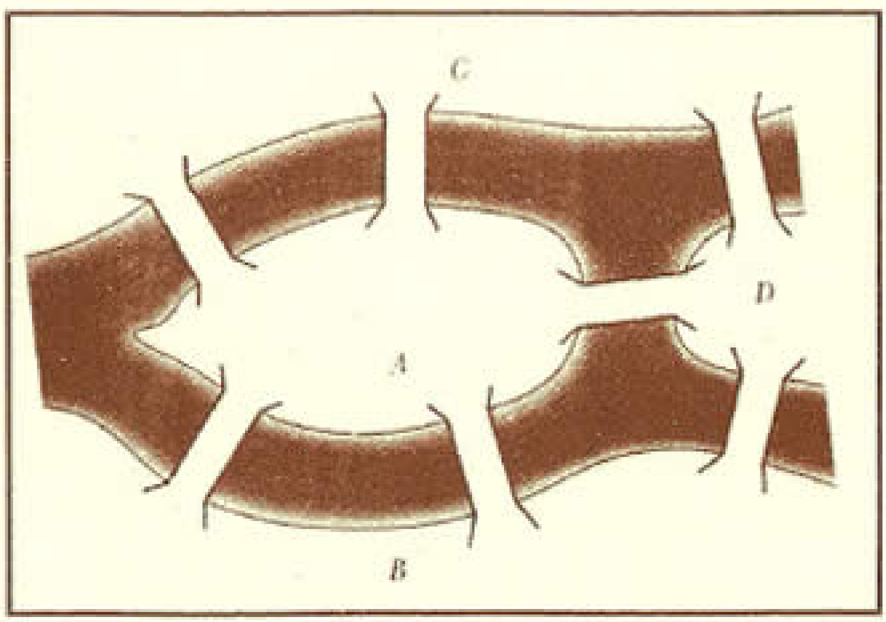
Discussion

If you have proved the case for n = 3 and n = 4, what other cases have you proved as well?

What kind of numbers do you need to prove for all cases ? (prime numbers)

What is the difficulty? (infinite number of primes)

Individual activity:



Konigsberg Bridge Networks

The river Pregel divides the town of Konigsberg into four separate parts, A,B,C, and D. Seven bridges connect the various parts of the town. A local riddle asked if it was possible to make a journey through all sections of the town such that each bridge is crossed once and only once.

How would you work this out?

Solution

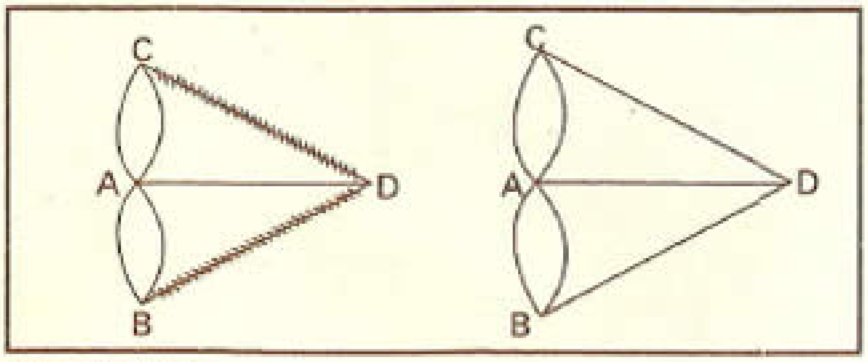


Diagram 4

Euler drew a network to represent the problem. He then argued that a point should be connected to an even number of lines, one for entering the land mass and one for leaving it. The only exception is when a traveler begins or ends a journey. If the point of entry and exit are different then those two points will have an odd number of lines, otherwise if the entry and exit point is the same then all points will have an even number of lines.

It turns out that Konigsberg is impossible to traverse because there are more than two points (land mass) with an odd number of lines (bridges).

Extension activities

- Traversibility problems

Which of the following figures can you draw without lifting your pencil and going over the same line twice. (They are called traversible figures)

Write down the number of even and odd nodes for each figure. What do you notice?

- Construct some solids. Can you discover the relationship found by Euler connecting Vertices, Edges and Faces?
- Research the number line and completeness of numbers. What are complex numbers?
- Research on prime numbers and encryption techniques.

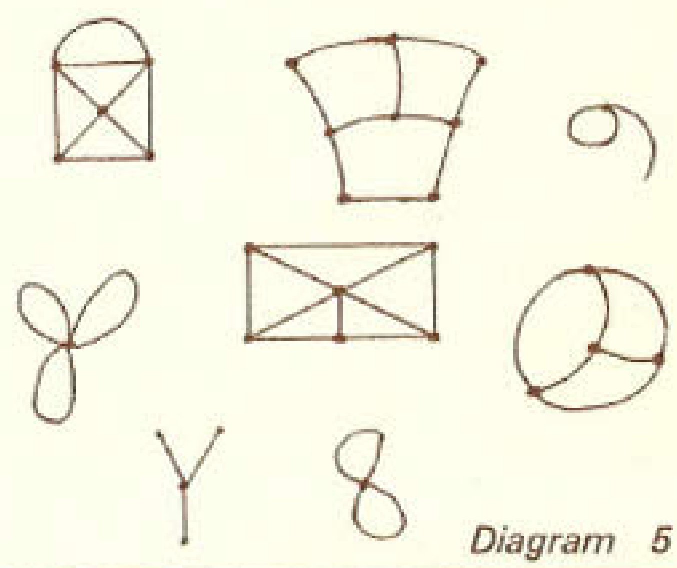


Diagram 5

Suggestions for rounding up the series of lessons

look at...

Look at the role of Elliptic equations and modulo mathematics in proving Fermat's theorem.

read about...

The man who solved the riddle was Andrew Wiles. Read about his motives, inspiration, turmoil etc. Follow his journey from his childhood dream to the final proof.

find out...

Find out how Cauchy and Lame stumbled over the unique factorization theorem—how an embedded truth in number theory got unstuck with the discovery of complex numbers, which mathematicians failed to notice for almost 200 years. Understand how mathematical proofs apply in a narrow range of well-defined parameters and become fallible when the parameters are changed.

explore...

Explore your approach to mathematics. Identify the qualities of those who excel in this field.

- Read this account of Andrew Wiles describing the experience of doing mathematics.

'Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room and it's dark, really dark, and one stumbles around bumping into the furniture. Gradually you learn where each piece of furniture is, and after six months or so, you find the light and suddenly it's all illuminated and you can see exactly where you are.'

— Andrew Wiles

Write an account of your experience of doing mathematics.

This is a proof for a method of finding all primitive pythagorean triples. There are sixteen lettered statements and you are asked to give reasons why they are true. Remember: Every lettered statement needs to be proved.

A Finding all primitive Pythagorean triples gives us all Pythagorean triples

$\Rightarrow$ We only need to consider primitive triples (x, y, z)

B x, y, z cannot all be odd

C x, y, z cannot be two even numbers and one odd number

D x, y, z cannot all be even

$\Rightarrow$ x, y, z consists of one even number and two odd numbers

E z is not the even number (Hint: Divisibility by 4)

$\Rightarrow$ either x or y is even

F We can assume x is the even number

G We can write $x^2 = (z + y)(z - y)$ instead of $x^2 + y^2 = z^2$

H $x, z + y, z - y$ are all even

So we can write $x = 2u, z + y = 2v, z - y = 2w$, where u, v, and w are positive whole numbers

I $u^2 = vw, z = v + w, y = v - w$

J v and w have no common factor (Hint: assume the contrary)

K $v\ '\ w = u^2$ tell us that v and w are both square numbers

$\Rightarrow$ We can find positive whole numbers p and q so that $p^2 = v$ and $q^2 = w$

L p and q have no common factor

M $z = p^2 + q^2 \quad y = p^2 - q^2$

N $x = 2pq$

O $p > q$

P p and q have different parity (That is if one is an odd number then the other is an even number)

All primitive pythagorean triples are of the form $(2pq, p^2-q^2, p^2+q^2)$ where

(i) $p>q$; (ii) p and q have different parity

Substituting values for p and q find some primitive pythagorean triples

To what values of p and q does the triple (4961, 6480, 8161) correspond?

This issue of Teacher's File has been contributed by Tanuj Shah